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Chaos in temperature in diluted mean-field spin-glass

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Abstract

We consider the problem of temperature chaos in mean-field spin-glass models defined on random lattices with finite connectivity. By means of an expansion in the order parameter we show that these models display a much stronger chaos effect than the fully connected Sherrington–Kirkpatrick model with the exception of the Bethe lattice with a bimodal distribution of the couplings.

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1. Introduction

Chaos in temperature is a very old problem in spin-glass theory and has received a lot of attention over the years in connection with many different problems. It has been studied by various approaches including scaling arguments and real space renormalization group analysis [1], analytical and numerical studies on mean-field models [2–6] (see also [7] for a recent review), analytical and numerical studies on the finite-dimensional Edwards–Anderson model [8–10], analytical and numerical studies on elastic manifolds in random media [11]. The problem of the temperature dependence of the Gibbs measure of the random energy model has also been investigated both in the physics literature [18] and in the mathematical physics literature [19]. It is also believed that the chaos picture is suitable to understand the surprising rejuvenation and memory effects observed in the dynamics of real spin-glasses, see for example [12–17] and references therein. Furthermore, a detailed understanding of this problem would probably shed some light on the success of the parallel tempering procedure, which is nowadays considered an essential ingredient to achieve thermalization in numerical spin-glass simulations [21, 22].

In this paper we consider the problem of chaos in temperature in various mean-field spin-glass models with finite connectivity. We show that these models display a much more pronounced chaos effect with respect to the fully connected Sherrington–Kirkpatrick (SK) model, with the notable exception of Bethe lattices with a bimodal distribution of the couplings; moreover we will analyze the dependence of chaos on various parameters. Chaos is much more pronounced if the system is locally heterogenous. It is possible that some of these results also hold in finite-dimensional models.

2. Some definitions

In the problem of temperature chaos one is interested in the correlations between the thermodynamically relevant configurations at different temperatures for a given general spin-glass model defined by a random Hamiltonian of N spins $H_J\{\sigma\}$. In particular, one would like to evaluate the probability $P_J^{\beta_1\beta_2}(q)$ of observing an overlap q if we extract two configurations according to their Boltzmann weights from systems with the same quenched Hamiltonian but different temperatures:

$$P_J^{\beta_1\beta_2}(q) = \frac{\overline{\sum_{\{\tau\}\{\sigma\}} \delta(Nq - \sum_i \sigma_i \tau_i) \exp[-\beta_1 H_J\{\sigma\} - \beta_2 H_J\{\tau\}]}}{\overline{\sum_{\{\tau\}\{\sigma\}} \exp[-\beta_1 H_J\{\sigma\} - \beta_2 H_J\{\tau\}]}} \quad (1)$$

where $\beta_i \equiv T_i^{-1}$ and the overline represents average with respect to different Hamiltonians $H_J\{\tau\}$.

A peculiar feature of replica-symmetry-breaking (RSB) theory [20] is that if the two systems have the same temperature, the function $P_J^{T_1 T_1}(q)$ in the low-temperature phase has a support between $-q_{EA}$ and q_{EA} , where q_{EA} is the so-called Edwards–Anderson parameter (for a two-spin interaction in the absence of a magnetic field). In particular, the disorder average $P(q) = \overline{P_J(q)}$ is given by $P(q) = dx/dq$ where $q(x)$ is a continuous function between zero and q_{EA} .

The problem of chaos in temperature concerns the function $P_J^{\beta_1\beta_2}(q)$; in particular, we say that there is chaos if

$$P_J^{\beta_1\beta_2}(q) = \delta(q), \quad (2)$$

i.e. if $P_J^{\beta_1\beta_2}(q)$ has a support concentrated on $q = 0$ and that there is no chaos otherwise. In the nutshell, if chaos is present, the equilibrium configurations at one temperature are quite different from those at a different temperature. It is clear that chaos may have a dramatic effect of the dynamics after a temperature shift.

The problem of chaos is intrinsically related to the disordered nature of these systems, being trivial in non-disordered models like a ferromagnet that have a translational invariant order parameter. The problem of chaos in temperature in the various mean-field models has been investigated intensively over the years. Today we know that there are full-RSB models that do not display chaos in temperature [4] and full-RSB models that do have chaos including notably the SK model [5]. Similarly there are 1RSB models that do have chaos in temperature and models that do not [6].

If chaos is present, we would like to quantify it, also to understand the finite volume effects (or finite time effects in the dynamics). More precisely we would like to know the free energy increase that happens if we constrain one system near the other (or more generally at an overlap q). This free energy increase can (as usually) also be written as a large deviation function for the distribution of the overlap between systems at different temperatures: it can be computed by studying two coupled systems.

The free energy of two systems forced to stay at a fixed overlap q is given by

$$F_{12}(q, \beta_1, \beta_2) = -\frac{1}{N} \ln \overline{\sum_{\{\tau\}\{\sigma\}} \delta\left(Nq - \sum_i \sigma_i \tau_i\right) \exp[-\beta_1 H_J\{\sigma\} - \beta_2 H_J\{\tau\}]}.$$

The function $F_{12}(q, \beta_1, \beta_2)$ must be larger than or equal to the free energies of the two unconstrained systems and the relevant quantity is the free energy shift $\Delta F_{12}(q, \beta_1, \beta_2) = F_{12}(q, \beta_1, \beta_2) - F(\beta_1) - F(\beta_2)$. Indeed if this quantity is greater than zero, it follows that the large deviations of the overlap are given by

$$P_J^{\beta_1\beta_2}(q) \propto \exp[-N \Delta F_{12}(q, \beta_1, \beta_2)]. \quad (3)$$

If the function $P_J^{\beta_1\beta_2}(q)$ has support on some non-zero values of q , then the free energy shift must vanish $\Delta F_{12}(q, \beta_1, \beta_2) = 0$. The opposite in general is not true, i.e. a vanishing free energy difference does not necessarily imply a non-zero $P_J^{\beta_1\beta_2}(q)$ as was unexpectedly discovered in the case $\beta_1 = \beta_2$ for the spherical SK model [7, 30].

In the following we will consider the constrained free energy functional $F_{12}(q, \beta_1, \beta_2)$ averaged over the disorder because it is usually assumed that this quantity (and correspondingly the large deviations) does not fluctuate in the large N limit. We could also define $F_{12}^Q(q, \beta_1, \beta_2)$ the free energy of the replica σ if we constrain the replica σ to stay at a fixed overlap from the replica τ , the replica τ being at equilibrium [27]:

$$F_{12}^Q(q, \beta_1, \beta_2) = -\frac{1}{N} \frac{\overline{\sum_{\{\tau\}} \exp[-\beta_2 H_J\{\tau\}] \ln \sum_{\{\sigma\}} \delta(Nq - \sum_i \sigma_i \tau_i) \exp[-\beta_1 H_J\{\sigma\}]}}{\sum_{\{\tau\}} \exp[-\beta_2 H_J\{\tau\}]}$$

In the first definition everything was symmetric in the two replicas and forcing the system to have a non-zero overlap we push out of equilibrium *both* replicas. In contrast in the second definition we look at the probability of σ , when it is constrained to stay at a fixed overlap with τ being a quenched configuration at equilibrium. It is evident that convexity implies that

$$F_{12}(q, \beta_1, \beta_2) < F_{12}^Q(q, \beta_1, \beta_2),$$

so that if $F_{12}(q, \beta_1, \beta_2)$ displays chaos, chaos is present also in $F_{12}^Q(q, \beta_1, \beta_2)$. It is interesting that we can obtain the internal energy as a function of q by performing a derivative with respect to β_2 . In the presence of chaos the quantity

$$\Delta E(\beta_1, \beta_2) = E_{12}^Q(q_{EA}, \beta_1, \beta_2) - F_{12}^Q(0, \beta_1, \beta_2) \tag{4}$$

should have the meaning of the energy that is *slowly* released after a sudden quench from a high temperature. Unfortunately these slow relaxations are too small to be observed experimentally.

In this paper we compute $F_{12}(q, \beta_1, \beta_2)$; the computation of $F_{12}^Q(q, \beta_1, \beta_2)$ could be done in a similar way: the two functions display a similar qualitative behavior.

In order to compute the free energy shift $\Delta F_{12}(q, \beta_1, \beta_2)$ it is convenient to consider the following *coupled* free energy:

$$\tilde{F}_{12}(\epsilon, \beta_1, \beta_2) = -\frac{1}{N} \ln \sum_{\{\tau\}\{\sigma\}} \exp \left[-\beta_1 H_J\{\sigma\} - \beta_2 H_J\{\tau\} + \epsilon \sum_{i=1}^N \sigma_i \tau_i \right], \tag{5}$$

where β_i is the inverse temperature of the corresponding system and N is the total number of spins in the system. When the coupling term ϵ vanishes, the above quantity is simply the sum of the average free energies (times β) at inverse temperatures β_1 and β_2 ; therefore, the relevant physical information concerning chaos is given by the difference $\Delta \tilde{F}_{12}(\epsilon, \beta_1, \beta_2) = \tilde{F}_{12}(\epsilon, \beta_1, \beta_2) - F_1(\beta_1) - F_2(\beta_2)$. In the thermodynamic limit, the two functions $F_{12}(q, \beta_1, \beta_2)$ and $\tilde{F}_{12}(\epsilon, \beta_1, \beta_2)$ become the Legendre transform of each other through the following relationships:

$$\tilde{F}_{12}(\epsilon) = F_{12}(q^*) - q\epsilon, \quad \left. \frac{dF}{dq} \right|_{q=q^*} = \epsilon, \tag{6}$$

and

$$F_{12}(q) = \tilde{F}_{12}(\epsilon^*) + q\epsilon, \quad -\left. \frac{d\tilde{F}}{d\epsilon} \right|_{\epsilon=\epsilon^*} = q. \tag{7}$$

3. Chaos in temperature in the generalized SK model

In this section we consider the problem of chaos in temperature in the context of the generalized SK model, that is a spin-glass model whose free energy is given by the extremization of a functional of an $n \times n$ matrix Q_{ab} with generic coefficients. Over the years many different spin-glass models have been mapped over a generalized Sherrington–Kirkpatrick model, e.g. the Edwards–Anderson model in an expansion in large dimension [28] and at a fixed dimension in the loop expansion above the upper critical dimension $D_u = 6$ [29]. In particular, such a mapping was used by Kondor to assess chaos in temperature in the Edwards–Anderson model [9, 10]. In the next section we will use it to study spin-glass models on the Bethe lattice.

In all the aforementioned spin-glass models $F_{12}(\epsilon, \beta_1, \beta_2)$ can be computed in the replica framework (see e.g. [7]), i.e. considering a set of n replicas of the two coupled systems and then sending n to zero. By means of standard manipulations one obtains the free energy as a functional $F_{12}(\hat{Q}, \epsilon)$ over a $2n \times 2n$ matrix $\hat{Q} = \begin{pmatrix} Q_1 & P \\ P^t & Q_2 \end{pmatrix}$, where Q_1, Q_2 and P are $n \times n$ matrices. The constrained free energy is obtained by extremizing the functional with respect to the order parameter \hat{Q} at a given value of ϵ . The functional may be very complicated, but one can obtain a more tractable expression by expanding it in powers of \hat{Q} . This expansion is perturbative near the critical temperature where \hat{Q} is expected to be small. In the end one obtains the following variational expression:

$$F_{12}(\hat{Q}, \epsilon) = F_{\text{para}}(\beta_1) + F_{\text{para}}(\beta_2) - \frac{\tau_1}{2} \text{Tr} Q_1^2 - \frac{\tau_2}{2} \text{Tr} Q_2^2 - \tau_{12} \text{Tr} P^2 - \frac{\omega}{6} \text{Tr} \hat{Q}^3 - \frac{v}{8} \text{Tr} \hat{Q}^4 + \frac{y}{4} \sum_{abc} \hat{Q}_{ab}^2 \hat{Q}_{ac}^2 - \frac{u}{12} \sum_{ab} \hat{Q}_{ab}^4 - \epsilon \sum_{a=1}^n P_{aa} + O(\hat{Q}^5), \quad (8)$$

where $F_{\text{para}}(\beta_1)$ and $F_{\text{para}}(\beta_2)$ are the terms that do not depend on \hat{Q} and are irrelevant for the present discussion. The above expression is $O(n)$ and we have neglected terms $O(n^2)$ that can be present and are relevant to evaluate the free energy fluctuations [24]. For the time being we assume that the only dependence on the temperature is in the reduced temperatures τ_1, τ_2 and τ_{12} while the other coefficients do not change with the temperature.

The variational action in the presence of a forcing term ϵ has been computed in [25] and will also be reconsidered in the appendix. It turns out that the chaos effect depends crucially on the parameters ω, v and c_{12} , that is the coefficient of the term $(\tau_1 - \tau_2)^2$ in the expansion of τ_{12} :

$$\tau_{12} = \frac{1}{2}(\tau_1 + \tau_2) + \frac{c_{12}}{4}(\tau_1 - \tau_2)^2 + O(\tau^3). \quad (9)$$

Following [7, 25] we report the following value of the free energy shift at leading order in q and $(\tau_1 - \tau_2)$:

$$\Delta F_{12}(q) = A|q|^3(\tau_1 - \tau_2)^2 \quad A = \frac{u}{6\omega} \left(\frac{v}{\omega^2} - c_{12} \right), \quad (10)$$

The above expression holds when q is small but larger than $|\tau_1 - \tau_2|$; in the opposite situation ($q \ll |\tau_1 - \tau_2|$), as we will show in the appendix, we have at leading order

$$\Delta F_{12}(q) = Bq^2|\tau_1 - \tau_2|^3 \quad B = \frac{u^{1/2}}{\omega 2^{3/2} \pi} \left(\frac{v}{\omega^2} - c_{12} \right)^{3/2}. \quad (11)$$

A peculiar feature of the SK model is that the quantity $\left(\frac{v}{\omega^2} - c_{12} \right)$ vanishes because $\omega = v = c_{12} = 1$; therefore, chaos is not present at this order. In this case a more refined

computation is necessary [5] and it shows that relationship (10) must be replaced with the much smaller expression

$$\Delta F_{12}(q) = \frac{12}{35} |q|^7 \Delta T^2, \quad (12)$$

where ΔT is the difference between the two temperatures. As a consequence chaos in temperature in the SK model is exceedingly weak and it was not observed in numerical simulations up to quite large system sizes [3].

4. Chaos in temperature on Bethe lattice spin-glass models

4.1. Setting up the computation

In [24] we have obtained the mapping of the free energy of spin-glass models defined on Bethe lattices with finite connectivity on the action (8). In this section we will extend those results to study chaos in temperature in these models.

Extending the treatment of [26] to the case of two coupled systems we express the free energy as a variational functional of the order parameter $\rho(\{\sigma_1, \sigma_2\})$ that is a function defined on $2n$ Ising spins $\{\sigma_1\} \equiv \sigma_1^1, \dots, \sigma_1^n$ and $\{\sigma_2\} \equiv \sigma_2^1, \dots, \sigma_2^n$. The variational expression of the free energy reads

$$\begin{aligned} \tilde{F}_{12}(\epsilon, \beta_1, \beta_2) = & \frac{M}{n} \ln \text{Tr}_{\{\sigma_1, \sigma_2\}} \rho^{M+1} \{ \sigma_1, \sigma_2 \} e^{\epsilon \sum_{a=1}^n \sigma_1^a \sigma_2^a} \\ & - \frac{M+1}{2n} \ln \int \text{Tr}_{\{\sigma_1, \sigma_2\}} \text{Tr}_{\{\tau_1, \tau_2\}} \rho^M \{ \sigma_1, \sigma_2 \} \rho^M \{ \tau_1, \tau_2 \} \\ & \times \left\langle \exp \left[\beta_1 J \sum_{\alpha} \sigma_1^{\alpha} \tau_1^{\alpha} + \beta_2 J \sum_{\alpha} \sigma_2^{\alpha} \tau_2^{\alpha} + \epsilon \sum_{a=1}^n \sigma_1^a \sigma_2^a + \epsilon \sum_{a=1}^n \tau_1^a \tau_2^a \right] \right\rangle, \quad (13) \end{aligned}$$

where $M+1$ is the connectivity of the lattice and the square brackets mean average with respect to the distribution of J . The above expression has to be extremized with respect to $\rho\{\sigma_1, \sigma_2\}$. We note that it is invariant under a rescaling of $\rho\{\sigma_1, \sigma_2\}$ so that we can choose any normalization for it. If we normalize $\rho\{\sigma_1, \sigma_2\}$ to 1, the corresponding variational equation in terms of $\rho(\sigma)$ reads

$$\rho\{\sigma_1, \sigma_2\} = \frac{1}{\mathcal{N}} \text{Tr}_{\{\tau_1, \tau_2\}} \rho^M \{ \tau_1, \tau_2 \} \left\langle \exp \left[\beta_1 J \sum_{\alpha} \sigma_1^{\alpha} \tau_1^{\alpha} + \beta_2 J \sum_{\alpha} \sigma_2^{\alpha} \tau_2^{\alpha} + \epsilon \sum_{a=1}^n \tau_1^a \tau_2^a \right] \right\rangle, \quad (14)$$

where \mathcal{N} is a normalization constant.

In order to build an expansion in the order parameter we write

$$\rho\{\sigma_1, \sigma_2\} = \sum_{k_1=0, k_2=0}^n b_{k_1, k_2} \sum_{(\alpha_1 \dots \alpha_{k_1}), (\beta_1 \dots \beta_{k_2})} q_{\alpha_1 \dots \alpha_{k_1}, \beta_1 \dots \beta_{k_2}} \sigma_1^{\alpha_1} \dots \sigma_1^{\alpha_{k_1}} \sigma_2^{\beta_1} \dots \sigma_2^{\beta_{k_2}} \quad (15)$$

with

$$b_{k_1, k_2} \equiv (\cosh^n \beta_1 J \cosh^n \beta_2 J \tanh^{k_1} \beta_1 J \tanh^{k_2} \beta_2 J). \quad (16)$$

The variational equation (14) can now be written as equations for $q_{\alpha_1 \dots \alpha_{k_1}, \beta_1 \dots \beta_{k_2}}$:

$$q_{\alpha_1 \dots \alpha_{k_1}, \beta_1 \dots \beta_{k_2}} = \frac{\text{Tr}_{\{\tau_1, \tau_2\}} \tau_1^{\alpha_1} \dots \tau_1^{\alpha_{k_1}} \tau_2^{\beta_1} \dots \tau_2^{\beta_{k_2}} \rho^M \{ \tau_1, \tau_2 \} \exp \epsilon \sum_{a=1}^n \tau_1^a \tau_2^a}{\text{Tr}_{\{\tau_1, \tau_2\}} \rho^M \{ \tau_1, \tau_2 \} \exp \epsilon \sum_{a=1}^n \tau_1^a \tau_2^a}. \quad (17)$$

The rhs of the above equations can be expanded in powers of $q_{\alpha_1 \dots \alpha_{k_1}, \beta_1 \dots \beta_{k_2}}$.

In general, the equations for two-index q_{ab} depend on higher order objects and this fact leads to rather complex equations. However near the critical temperature Q with a large number

of index terms are much smaller than the one with two indices and using the corresponding variational equations they can be eliminated [23]. In this way we can obtain an expression that depends only on a $2n \times 2n$ matrix \hat{Q} like the one appearing in the variational action (8). This was done in [24] for the single system and we have extended that computation to the coupled system case.

If we start from the equations in appendix B of [24], we divide all the equations by a factor 2 and we make a rescaling $Q_1 \rightarrow Q_1/(b_{11}(M - 1))$, $Q_2 \rightarrow Q_2/(b_{22}(M - 1))$, $P \rightarrow P/(b_{12}(M - 1))$, we obtain that equations (17) for \hat{Q} are the same as those that would be obtained from a variational action of the form (8) with the following coefficients:

$$\tilde{\tau}_1 = \frac{Mb_{11} - 1}{2b_{11}(M - 1)} \tag{18}$$

$$\tilde{\tau}_2 = \frac{Mb_{22} - 1}{2b_{22}(M - 1)} \tag{19}$$

$$\tilde{\tau}_{12} = \frac{Mb_{12} - 1}{2b_{12}(M - 1)} \tag{20}$$

$$\tilde{\omega} = \frac{M}{M - 1} \tag{21}$$

$$\tilde{v} = \frac{M(Mb_4 + M - 2)}{(1 - Mb_4)(M - 1)^2} \tag{22}$$

$$\tilde{u} = \frac{M(M(2M - 1)b_4 + M - 2)}{(1 - Mb_4)(M - 1)^2} \tag{23}$$

where $b_4 = \langle (\tanh \beta_c J)^4 \rangle$ and β_c is the inverse critical temperature that obeys the equation $1 = M \langle \tanh^2 \beta_c J \rangle$. The mixed reduced temperature $\tilde{\tau}_{12}$ is such that its expansion in terms of $\tilde{\tau}_1$ and $\tilde{\tau}_2$ is of the form (9) with the following expression for c_{12} :

$$\tilde{c}_{12} = \frac{2(M - 1)}{M} - \frac{(M - 1)J^2(1 - \tanh^2 \beta_c J)^2}{(J \tanh \beta_c J - J \tanh^3 \beta_c J)^2 M^2}. \tag{24}$$

The SK limit is recovered by sending M to infinity and the coupling strength to zero as $J^2 = 1/M$. In this limit we have $b_{ij} = 1/(MT_i T_j)$, the critical temperature goes to 1 and the various coefficients read

$$\tilde{\tau}_1 = \frac{1 - T_1^2}{2}, \quad \tilde{\tau}_2 = \frac{1 - T_2^2}{2}, \quad \tilde{\tau}_{12} = \frac{1 - T_1 T_2}{2}, \tag{25}$$

$$\tilde{u} = \tilde{\omega} = \tilde{v} = \tilde{c}_{12} = 1. \tag{26}$$

These are precisely the coefficients obtained for the SK model, see e.g. [25]. The rescaling $Q_1 \rightarrow Q_1/(b_{11}(M - 1))$, $Q_2 \rightarrow Q_2/(b_{22}(M - 1))$, $P \rightarrow P/(b_{12}(M - 1))$ was performed in order to get rid of the temperature dependence in all coefficients other than $\tilde{\tau}_1$, $\tilde{\tau}_2$ and $\tilde{\tau}_{12}$ and corresponds in the SK limit to the usual rescaling $Q_1 \rightarrow Q_1/\beta_1^2$, $Q_2 \rightarrow Q_2/\beta_2^2$, $P \rightarrow P/(\beta_1 \beta_2)$. The above definitions are such that at finite M the reduced temperature goes to 1/2 at zero temperature. The actual dependence of the reduced temperature with respect to the temperature is such that

$$\tilde{\tau} = \frac{(J \tanh \beta_c J - J \tanh^3 \beta_c J)M^2}{(M - 1)T_c^2} (T_c - T) + O(T_c - T)^2 \tag{27}$$

and the prefactor goes to 1 in the SK limit.

The above coefficients however cannot be put simply into equations (10) and (11) in order to obtain the free energy shifts. We must bear in mind that once the variational equations (17) are expanded in powers of \hat{Q} they *look like* as if they were obtained from a variational free energy of the form (8) with coefficients that in order to avoid possible confusion we represent as tilded. This does not mean that the true free energy has an expansion with the same coefficients; as was shown in appendix B of [24], this can be understood noting that the equation for the order parameter corresponds to the following expression:

$$0 = \text{Tr} \left[\sigma_a \sigma_b \left(\rho(\sigma) - \frac{\text{Tr}_\tau \rho^M(\tau) \langle \exp J \sum_c \sigma_c \tau_c \rangle}{\text{Tr} \rho^M(\tau)} \right) \right] \quad (28)$$

while the equation one obtains by differentiating expression (13) corresponds to

$$0 = \text{Tr} \left[\rho^{M-1}(\{\sigma\}) \sigma_a \sigma_b \left(\rho(\sigma) - \frac{\text{Tr}_\tau \rho^M(\tau) \langle \exp J \sum_c \sigma_c \tau_c \rangle}{\text{Tr} \rho^M(\tau)} \right) \right]. \quad (29)$$

Thus the two expressions are equivalent in the sense that they have the same solution at the order at which they are valid. In order to obtain the expansions in powers of the order parameter matrix one could expand directly the free energy (13), but technically it is much simpler to expand the variational equations (17).

This problem can be bypassed noticing that the derivatives of expressions (10) and (11) *with the tilded coefficients* allow us to determine q as a function of ϵ . In the two different regimes we have

$$\frac{3q^2 \tilde{u}}{6\tilde{\omega}} \left(\frac{\tilde{v}}{\tilde{\omega}^2} - \tilde{c}_{12} \right) (\tilde{\tau}_1 - \tilde{\tau}_2)^2 = \epsilon \quad (30)$$

and

$$2q \frac{\tilde{u}^{1/2}}{\tilde{\omega} 2^{3/2} \pi} \left(\frac{\tilde{v}}{\tilde{\omega}^2} - \tilde{c}_{12} \right)^{3/2} |\tilde{\tau}_1 - \tilde{\tau}_2|^3 = \epsilon. \quad (31)$$

We must take into account that the overlap q appearing in the above equation is not the true overlap. This is due to two reasons.

- We must recall that we have carried out the rescaling $Q_1 \rightarrow Q_1/(b_{11}(M-1))$, $Q_2 \rightarrow Q_2/(b_{22}(M-1))$, $P \rightarrow P/(b_{12}(M-1))$.
- A more subtle reason is that the true overlap is given by

$$q = \frac{\text{Tr}_{\{\tau_1, \tau_2\}} \tau_1^1 \tau_2^1 \rho^{M+1} \{\tau_1, \tau_2\} \exp \epsilon \sum_{a=1}^n \tau_1^a \tau_2^a}{\text{Tr}_{\{\tau_1, \tau_2\}} \rho^{M+1} \{\tau_1, \tau_2\} \exp \epsilon \sum_{a=1}^n \tau_1^a \tau_2^a}. \quad (32)$$

The difference is that there is a term ρ^{M+1} while in equations (17) there is a power ρ^M . As usual the overlap entering in the cavity equations is not the true overlap.

The net effect is that in order to obtain *at leading order* the relationship between the true overlap and the forcing ϵ , we have to make the following rescaling in equations (30) and (31):

$$q \rightarrow \frac{b_{12}(M-1)}{1+b_{12}} q. \quad (33)$$

In the SK limit the above rescaling reduces to $q \rightarrow \beta_1 \beta_2 q$. Near the critical temperature we have

$$\frac{b_{12}(M-1)}{1+b_{12}} = \frac{M-1}{M+1}. \quad (34)$$

The corresponding expressions yield the overlap as a function of the forcing and can be integrated back to obtain the correct free energy shifts in the two regimes $(\tilde{\tau}_1 - \tilde{\tau}_2) \ll q$ and $q \ll (\tilde{\tau}_1 - \tilde{\tau}_2)$:

$$\Delta F_{12}(q) = A |q|^3 (\tilde{\tau}_1 - \tilde{\tau}_2)^2 \quad A = \left(\frac{M-1}{M+1} \right)^2 \frac{\tilde{u}}{6\tilde{\omega}} \left(\frac{\tilde{v}}{\tilde{\omega}^2} - \tilde{c}_{12} \right), \quad (35)$$

and

$$\Delta F_{12}(q) = B q^2 |\tilde{\tau}_1 - \tilde{\tau}_2|^3 \quad B = \left(\frac{M-1}{M+1} \right) \frac{\tilde{u}^{1/2}}{\tilde{\omega} 2^{3/2} \pi} \left(\frac{\tilde{v}}{\tilde{\omega}^2} - \tilde{c}_{12} \right)^{3/2}. \quad (36)$$

We can now apply the previous formulas to different distributions of J .

4.2. Diluted bimodal distribution

A surprising feature of the above expressions is that a direct computation shows that the quantity $\tilde{v}/\tilde{\omega}^2 - \tilde{c}_{12}$ vanishes in the case of a bimodal distribution of the coupling $J = \pm 1$ as in the SK model. Therefore for these models we expect chaos to be a much smaller effect possibly of the same order of the SK model. This is consistent with the fact that chaos is very difficult to be observed in numerical simulations of these models.

This can be seen by considering the case of the random-bond bimodal distribution where any coupling in the lattice is zero with the probability p or ± 1 with the probability $1 - p$:

$$P(J) = p\delta(J) + \frac{(1-p)}{2}(\delta(J+1) + \delta(J-1)) \quad (0 \leq p \leq 1). \quad (37)$$

In this case the relevant parameters to be inserted in equations (35) and (36) read

$$\beta_c = \operatorname{arctanh} \frac{1}{\sqrt{M(1-p)}}, \quad (38)$$

$$\tilde{\omega} = \frac{M}{M-1}, \quad \tilde{u} = \frac{M(1-M^2(1-p) - 2Mp)}{(M-1)^2(1-M(1-p))}, \quad (39)$$

$$|\tilde{\tau}_1 - \tilde{\tau}_2| = \frac{M(M(1-p) - 1) \operatorname{arctanh}^2 \frac{1}{\sqrt{M(1-p)}}}{(M-1)\sqrt{M(1-p)}} |T_1 - T_2| + O(T_1 - T_2)^2. \quad (40)$$

In the last expression we have used equation (27). The chaos prefactor is

$$\frac{\tilde{v}}{\tilde{\omega}^2} - \tilde{c}_{12} = \frac{p}{M(1-p) - 1} \quad (41)$$

and we see that it vanishes in the purely bimodal case corresponding to $p = 0$.

4.3. Poissonian distribution of the connectivity

The parameters computed above can also be used to study the model where the connectivity of each spin has a Poissonian distribution. In general we have to take the $M \rightarrow \infty$ limit in the above expressions sending $p \rightarrow 1$ as

$$p = 1 - \frac{\alpha}{M}, \quad (42)$$

where α is the average connectivity of a site. In the case where the coupling strength is ± 1 the relevant parameters read

$$\beta_c = \operatorname{arctanh} \frac{1}{\sqrt{\alpha}} \quad (43)$$

$$\tilde{\omega} = 1 \quad (44)$$

$$\tilde{u} = \frac{2 + \alpha}{\alpha - 1} \quad (45)$$

$$\frac{\tilde{v}}{\tilde{\omega}^2} - \tilde{c}_{12} = \frac{1}{\alpha - 1} \quad (46)$$

$$|\tilde{\tau}_1 - \tilde{\tau}_2| = \frac{\alpha - 1}{\sqrt{\alpha}} \operatorname{arctanh}^2 \frac{1}{\sqrt{\alpha}} |T_1 - T_2| + O(T_1 - T_2)^2. \quad (47)$$

$$\frac{\tilde{v}}{\tilde{\omega}^2} - \tilde{c}_{12} = \frac{1}{\alpha - 1}. \quad (48)$$

The free energy differences read

$$\Delta F_{12}(q) = |q|^3 \frac{2 + \alpha}{6\alpha} \operatorname{arctanh}^2 \frac{1}{\sqrt{\alpha}} \left(\frac{T_1 - T_2}{T_c} \right)^2 \quad (49)$$

$$\Delta F_{12}(q) = q^2 \frac{(2 + \alpha)^{1/2} (\alpha - 1)}{(2\alpha)^{3/2} \pi} \operatorname{arctanh}^4 \frac{1}{\sqrt{\alpha}} \left| \frac{T_1 - T_2}{T_c} \right|^3. \quad (50)$$

5. Conclusions

We have shown that mean-field spin-glass models defined on random lattices with finite connectivity display chaos in temperature. Chaos is stronger in perturbation theory than in the SK model by four orders of magnitude.

On general grounds for small values of the overlaps q we could expect a chaos effect, such as $\Delta F_{12}(q) \propto q^2 |T_1 - T_2|^2$, instead as we have seen in section (3) the effect is smaller $\Delta F_{12}(q) \propto q^2 |T_1 - T_2|^3$. In other words, chaos in the generalized SK model, to which the random lattice models can be mapped, is larger than in SK but is nevertheless not as strong as one could naively expect. This is because in Bethe lattice models chaos is truly a RSB effect [6]: a model with a stable RS phase has a single stable thermodynamic state that can be followed by increasing or decreasing the temperature and therefore is not chaotic³. The connection between chaos and RSB is reflected by the fact that the coefficients of the free energy shifts (equations (10) and (11)) depend on the coefficient u of the quartic interaction that is responsible for RSB.

In diluted models chaos in temperature is considerably stronger than in SK with the notable exception of Bethe lattices with bimodal interactions as shown by equation (41). In this case we expect the effect to be of the same order of magnitude of the SK model. In the SK model one can prove that the quadratic terms in q^2 in the free energy shift vanishes at all orders [9, 25], and we mention that the same result can be proved at all orders in the case of the Bethe lattice with a purely bimodal interaction, but this will be published elsewhere. The argument relies on the local homogeneity of the Bethe lattice with bimodal interactions that is also responsible for non-Gaussian free energy fluctuations [24, 32].

The fact that chaos in temperature on Bethe lattice with a bimodal interaction is as weak as in the SK model is also supported by the existing numerical results. The numerical data of

³ The spherical SK model is RS and is indeed non-chaotic but has a zero free energy shift at order N [7, 30]. However, this is a consequence of the fact that the model is marginally stable in the whole low-temperature phase. A different scenario is present in the Migdal–Kadanoff approximation [1]; however, the presence of a few spins with a very large number of connections may have deep effects on the properties of this model.

Billoire and Marinari (BM) suggest the absence of chaos in SK [3]; only a close look at the function $P^{\beta_1 \beta_2}(q)$ gives a hint that the effect may be present due to a very slow increase of the weight in $P^{\beta_1 \beta_2}(0)$ with the system size. On the other hand, the theoretical value computed in [5] shows that the effect is exceedingly small in the SK model and that it is practically unobservable at the system sizes simulated in [3]. BM also considered the Bethe lattice with connectivity $c = 6$ and bimodal distribution of the coupling finding again no strong chaos effect in agreement with the results presented here.

Finally we would like to remark that the extension of our computations to finite-dimensional models (at least for large dimensions) can be carried out using the techniques of [28]. It would be very interesting to study if we can obtain reliable predictions in high-dimensional models, e.g. $D = 6$.

Appendix. The quadratic action

In this appendix we compute the free energy shift when ϵ is much smaller than any other parameter in the theory. In this case the problem can be solved by expanding the variational action (8) at second order around the solution $P_{ab} = 0$. A similar treatment was recently put forward in the case of bond chaos [7, 31]. The P -dependent term in the action can be written as

$$F_{12}(P, \epsilon) = \frac{1}{2} \text{Tr} P A P - \epsilon \text{Tr} P + o(P^2) \quad (\text{A.1})$$

where the matrix A is given by

$$A_{ab} = - \left(2\tau_{12} - \frac{y}{n} \text{Tr} Q_1^2 - \frac{y}{n} \text{Tr} Q_2^2 \right) \delta_{ab} - \omega(Q_1 + Q_2)_{ab} - v(Q_1^2 + Q_2^2 + Q_1 Q_2)_{ab} \quad (\text{A.2})$$

extremizing with respect to P the above expression at given ϵ we easily obtain

$$F_{12}(\epsilon, \beta_1, \beta_2) = -\frac{\epsilon^2}{2} \text{Tr} A^{-1}. \quad (\text{A.3})$$

In order to compute the trace we diagonalize the matrix A_{ab} . The eigenvalues of a hierarchical matrix described by $(a_d, a(x))$ are given by

$$\begin{aligned} \lambda_0 &= a_d - \int_n^1 a(x) dx & \text{deg} : 1 \\ \lambda_x &= a_d - \left(x a(x) + \int_x^1 q(x) dx \right) & \text{deg} : -n \frac{dx}{x^2} \end{aligned}$$

and the trace is given by

$$\lim_{n \rightarrow 0} \frac{1}{n} \sum_a \frac{1}{\lambda_a} = \frac{1}{\lambda_1} - \int_0^1 \dot{a}(y) \frac{1}{\lambda(y)^2}. \quad (\text{A.4})$$

Let us examine the eigenvalue $\lambda_a(0)$:

$$-\lambda_a(0) = 2\tau_{12} - \frac{y}{n} \text{Tr} Q_1^2 - \frac{y}{n} \text{Tr} Q_2^2 - \omega(\bar{q}_1 + \bar{q}_2) + v(\bar{q}_1^2 + \bar{q}_2^2 + \bar{q}_1 \bar{q}_2). \quad (\text{A.5})$$

At this point we exploit the fact that when $\tau_1 = \tau_2$, this eigenvalue must vanish obtaining the condition

$$\tau_1 - \frac{y}{n} \text{Tr} Q_1^2 - \omega \bar{q}_1 + \frac{3}{2} v \bar{q}_1^2 = 0. \quad (\text{A.6})$$

Summing the above equation for τ_1 and τ_2 to the expression of $\lambda_a(0)$ we obtain

$$\lambda_a(0) = -2\tau_{12} + \tau_1 + \tau_2 + \frac{v}{2} (\bar{q}_1 - \bar{q}_2)^2 = -\frac{1}{2} \left(c_{12} - \frac{v}{\omega^2} \right) (\tau_1 - \tau_2)^2. \quad (\text{A.7})$$

Thus we encounter the same factor $(c_{12} - \frac{v}{\omega^2})$ of expression (10). In order to complete the computation of the trace we need the expression of $\lambda_a(x)$ around $x = 0$. In order to do this we use the relationship $\dot{\lambda}_a(x) = -x\dot{a}(x)$, where the dot means derivative with respect to x . At leading order in τ_1 and τ_2 we have $a(x) = -\omega(q_1(x) + q_2(x))$. The solution of the free problem is such that $q_1(x) = q_2(x) = \omega x/2u$ [20]; therefore,

$$\lambda_a(x) = \lambda_a(0) + \frac{\omega^2}{2u}x^2 \longrightarrow (\lambda_a(x))^2 = \lambda_a^2(0) + \lambda_a(0)\frac{\omega^2}{u}x^2. \quad (\text{A.8})$$

Thus at leading order for small $\lambda_a(0)$ we have

$$\lim_{n \rightarrow 0} \frac{1}{n} \text{Tr} A^{-1} = \int_0^1 \frac{\omega^2}{u} \frac{dx}{\lambda_a^2(0) + \lambda_a(0)\frac{\omega^2}{u}x^2} \simeq \frac{\omega}{u^{1/2}\lambda_a^{3/2}(0)} \int_0^\infty \frac{du}{1+y^2} = \frac{\omega\pi}{2u^{1/2}\lambda_a^{3/2}(0)}; \quad (\text{A.9})$$

therefore

$$\Delta F_{12}(\epsilon, \beta_1, \beta_2) = -\frac{\omega\epsilon^2\pi}{4u^{1/2}\lambda_a^{3/2}(0)} \quad (\text{A.10})$$

and

$$\Delta F_{12}(q, \beta_1, \beta_2) = q^2 \frac{u^{1/2}}{\omega 2^{3/2}\pi} \left(c_{12} - \frac{v}{\omega^2}\right)^{3/2} |\tau_1 - \tau_2|^3. \quad (\text{A.11})$$

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